Fast multiple precision $\exp(x)$ with precomputations

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Main challenges

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Given $x \in [0,1)$ and fixed $n \in \mathbb{N}$, compute $\exp x$ with a precision of n binary digits

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- Complexity as a function of n ?
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Focus on $\exp$, but similar approach applies to $\log$, $\sin$, $\cos$, etc.

Complexity measures

$n \longrightarrow$ bit precision

$\mathsf{A}(n)$	cost to add two n-bit numbers
$\mathsf{M}(n)$	cost to multiply two n-bit numbers
$\mathsf{P}(n)$	cost to compute a "smooth" n-bit product of many small numbers
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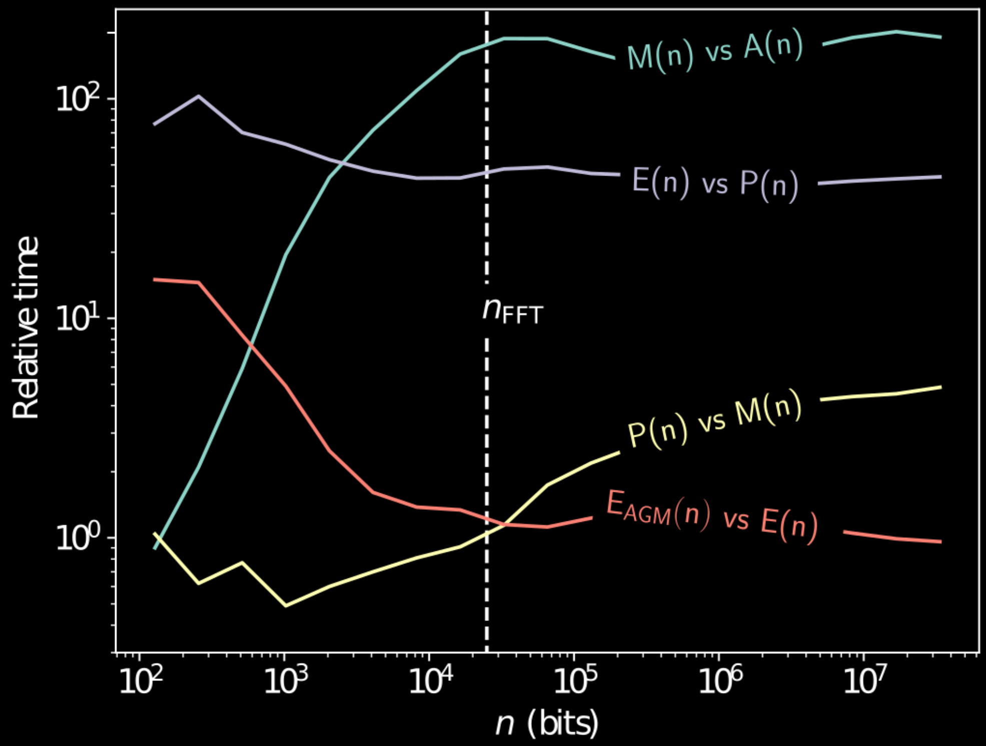
In theory:

$$\begin{aligned} \mathsf{A}(n) &= O(n) \\ \mathsf{M}(n) &= O(n \log n) \\ \mathsf{P}(n) &= O(\mathsf{M}(n) \log n) \\ \mathsf{E}(n) &= O(\mathsf{M}(n) \log n) \end{aligned}$$

Practical complexity, currently in FLINT



General ideas for exponentiation

Argument reduction

$$x = x_1 + \cdots + x_k + \varepsilon$$
$$e^x = e^{x_1} \times \cdots \times e^{x_k} \times e^{\varepsilon}$$

$\longrightarrow$ multiple decomposition strategies

Taylor series expansion

$$e^{\varepsilon} \approx 1 + \varepsilon + \frac{1}{2}\varepsilon^2 + \cdots + \frac{1}{(N-1)!}\varepsilon^{N-1}$$

$\longrightarrow$ multiple strategies to evaluate the Taylor series

Bitwise reduction (naive)

$$x = 0.\ 0101\ \ 1001\ \ 0011\ \ 1011\ \ \cdots$$

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$$
\begin{aligned}
x &= 0.\ 0101\ 1001\ 0011\ 1011\ \cdots \\
&= 0.\ 0101\ 0000\ 0000\ 0000\ \cdots\ + \\
&\quad\ 0.\ 0000\ 1001\ 0000\ 0000\ \cdots\ + \\
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&\quad\ \cdots
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{e}^x &= 1.\ 0101\ 1101\ 1110\ 1001\ \cdots\ \times \\
&\quad\ 1.\ 0000\ 1001\ 0010\ 1000\ \cdots\ \times \\
&\quad\ 1.\ 0000\ 0000\ 0011\ 0000\ \cdots\ \times \\
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Bitwise reduction (CORDIC, BKM)

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$$
x \approx 1.\,0110 \times 1.\,0000\,0111 \times 1.\,0000\,0000\,1100 \times 1.\,0000\,0000\,0000\,1101
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$$x \approx 1.\,0110 \times 1.\,0000\,0111 \times 1.\,0000\,0000\,1100 \times 1.\,0000\,0000\,0000\,1101$$

$\longrightarrow$ “Shift and add”

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$\longrightarrow$ "Shift and add" $\longrightarrow$ "Shift and FMA"

Complexity analysis

Reduction $x \longrightarrow \varepsilon$ with $\varepsilon < 2^{-r}$

m-bitwise

	Naive	**Shift and FMA**
Time	$\frac{r}{m}\mathsf{M}(n)$	$\frac{3r}{m}\mathsf{A}(n)$
Space	$\frac{2^m}{m}rn$	$\frac{2^m}{m}rn$

Diophantine approximation (Shönhage, Johansson)

$$x = 0.11062024$$

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$$x := 0.11062024 \approx 4\,\ell_3$$

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$$x - 4\,\ell_3 \approx 0.005351 \approx \ell_4$$

Diophantine approximation (Shönhage, Johansson)

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x \approx 4\,\ell_3 + \ell_4 - 26\,\ell_5 + 2\,\ell_6 + 6\,\ell_7
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$$\text{Time} \quad 1.4\, r\, \mathsf{A}(n) + \mathsf{P}(\overbrace{\mathrm{e}^3 m^2 \log_2 m}^{\leqslant n}) + \mathsf{M}(n)$$

$$\text{Space} \quad m\, n$$

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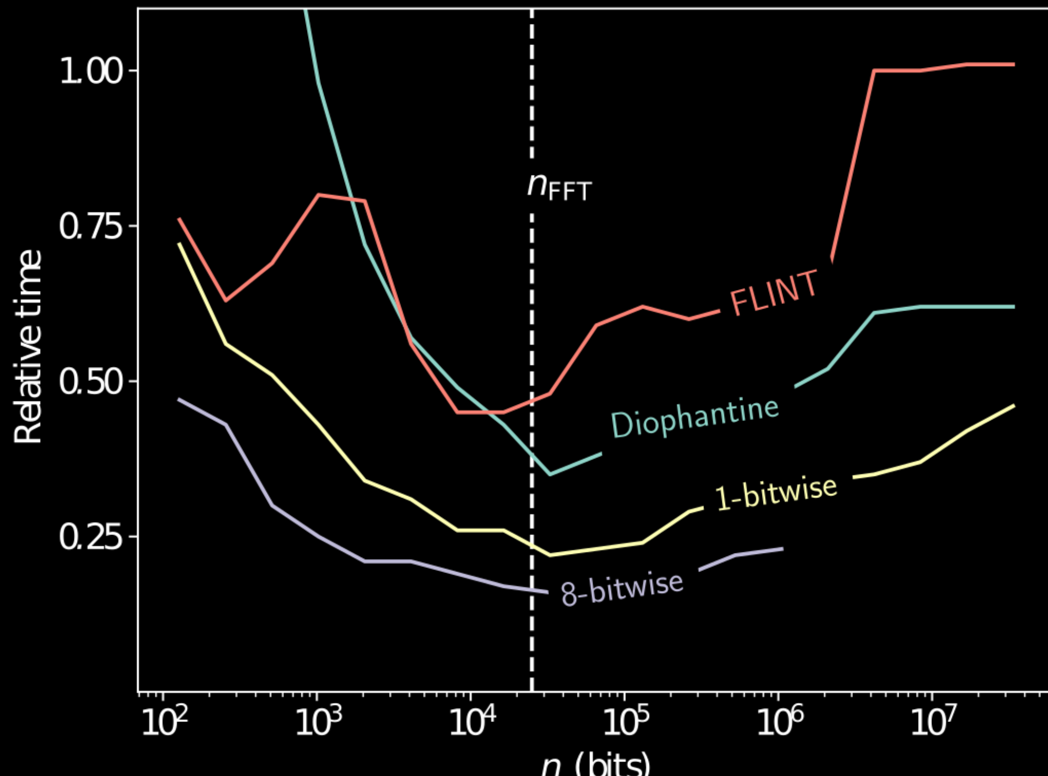
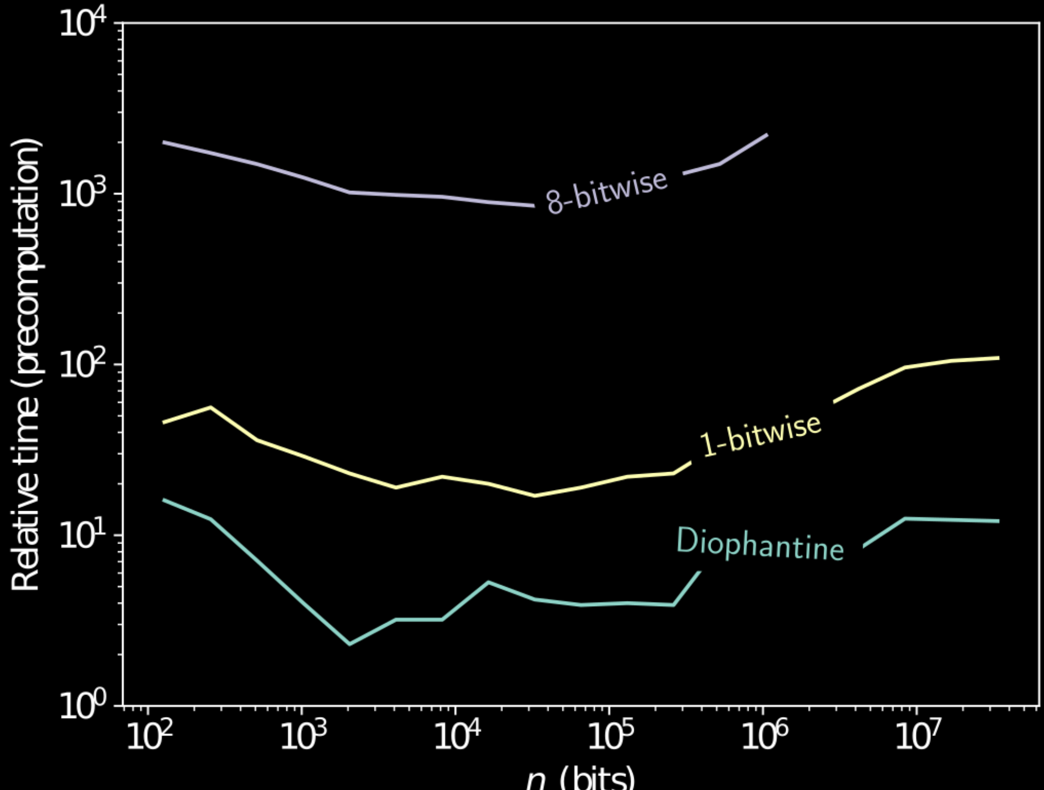
$$\text{Space} \quad m\, n$$

Optimum $\quad r \;\approx\; 2.9\, m \;\approx\; 0.64 \sqrt{\frac{n}{\log_2 n}}$

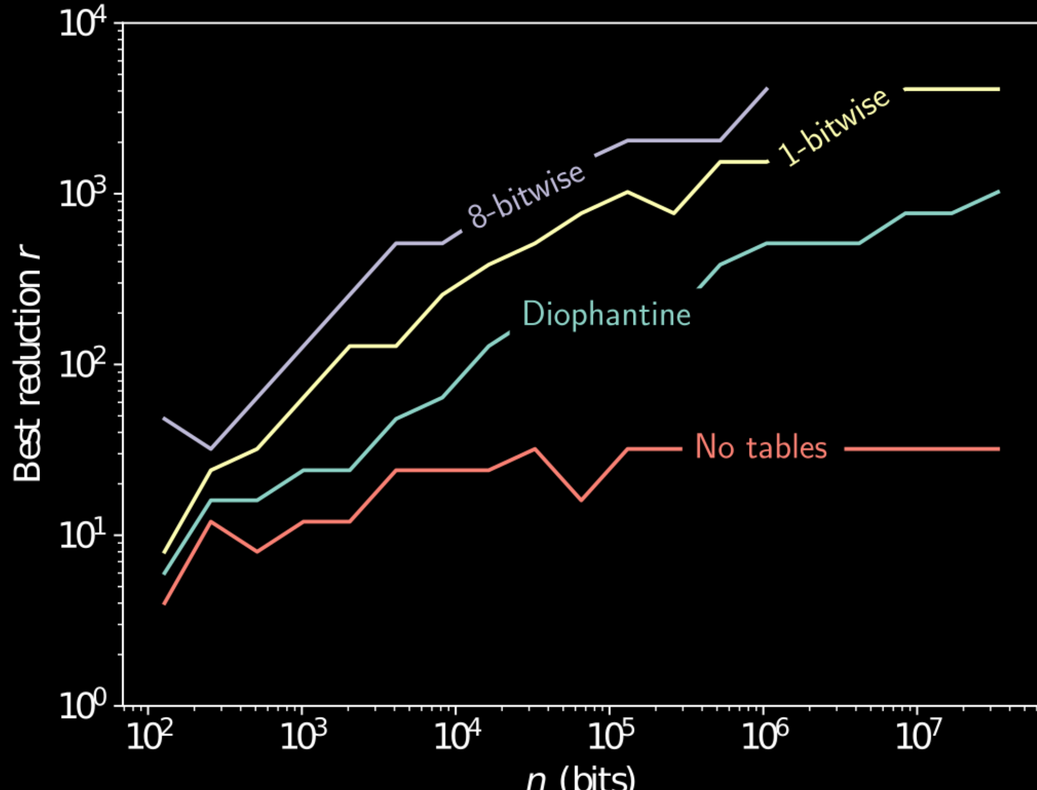
Overall timings w.r.t. squaring + Taylor series



Cost of precomputations



Best reduction threshold r



Taylor series

$$e^{\varepsilon} = 1 + \varepsilon + \frac{1}{2}\varepsilon^2 + \cdots + \frac{1}{(N-1)!}\varepsilon^{N-1}, \qquad N \approx \frac{n}{r}$$

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Horner's method $\frac{n}{r}\mathsf{M}(n)$

Hyperbolic sines $\frac{n}{2r}\mathsf{M}(n)$

$$\operatorname{sh}\varepsilon = \varepsilon + \tfrac{1}{6}\varepsilon^3 + \cdots + \tfrac{1}{(2N-1)!}\varepsilon^{2N-1}, \qquad N \approx \tfrac{n}{2r}$$

$$= \frac{e^{\varepsilon} - e^{-\varepsilon}}{2}$$

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$$e^{\varepsilon} = \operatorname{sh}\varepsilon + \sqrt{1 + \operatorname{sh}^2\varepsilon}$$

Taylor series

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Curiosity: generalization with asymptotic complexity $\frac{n}{3r}\mathsf{M}(n)$

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Rectangular splitting $c\sqrt{\frac{n}{r}}\mathrm{M}(n), \qquad \left(\tfrac{2}{3} \leqslant c \leqslant \tfrac{5}{3}\right)$

Binary splitting, bit burst

$$x = 0.\ 01\ 01\ 1011\ 01001001\ 0000111010111010\ 01001010010100111001010010110010$$

$$\varepsilon = 0.\ 00\ 00\ 0000\ 01001001$$

$$e^{\varepsilon} \approx 1 + \varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{6}\varepsilon^3 + \frac{1}{24}\varepsilon^4 + \frac{1}{120}\varepsilon^5 + \frac{1}{720}\varepsilon^6 + \frac{1}{5040}\varepsilon^7$$

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$1+\varepsilon$ $\quad 3+\varepsilon$ $\quad 5+\varepsilon$ $\quad 7+\varepsilon$ $\quad \approx$16 bits

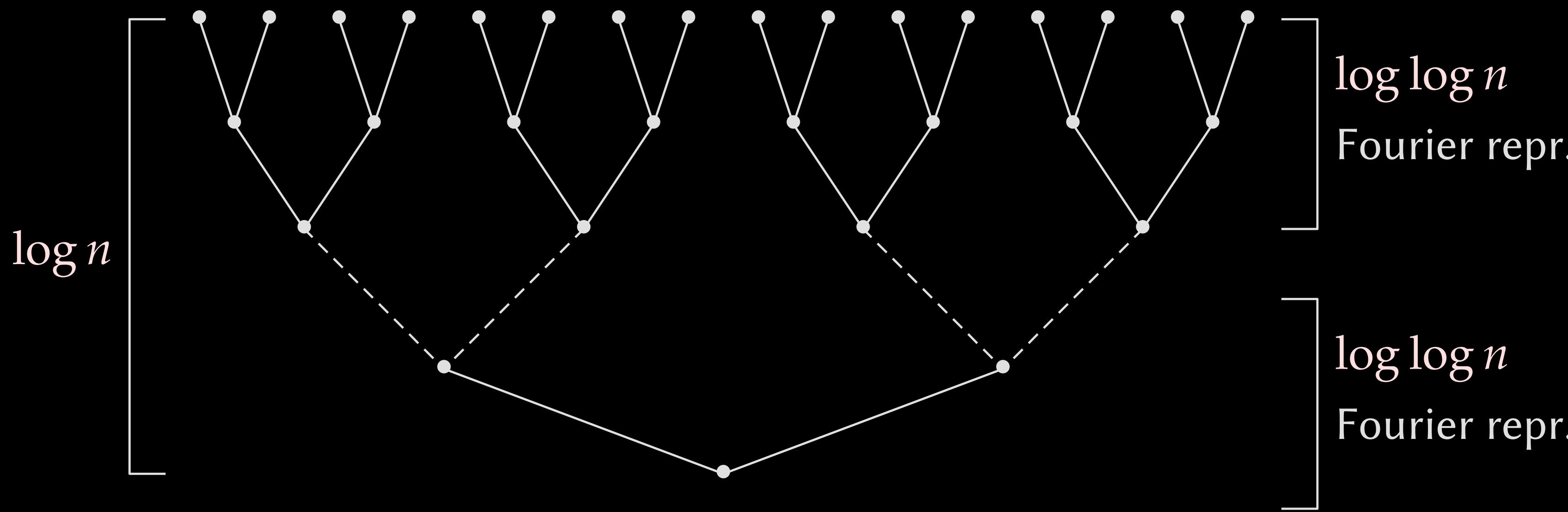
$\times 6$ $\quad \times\varepsilon^2$ $\quad \times 42$ $\quad \times\varepsilon^2$

$6+6\varepsilon+3\varepsilon^2+\varepsilon^3$ $\quad 210+42\varepsilon+7\varepsilon^2+\varepsilon^3$ $\quad \approx$32 bits

$\times 840$ $\quad \times\varepsilon^4$

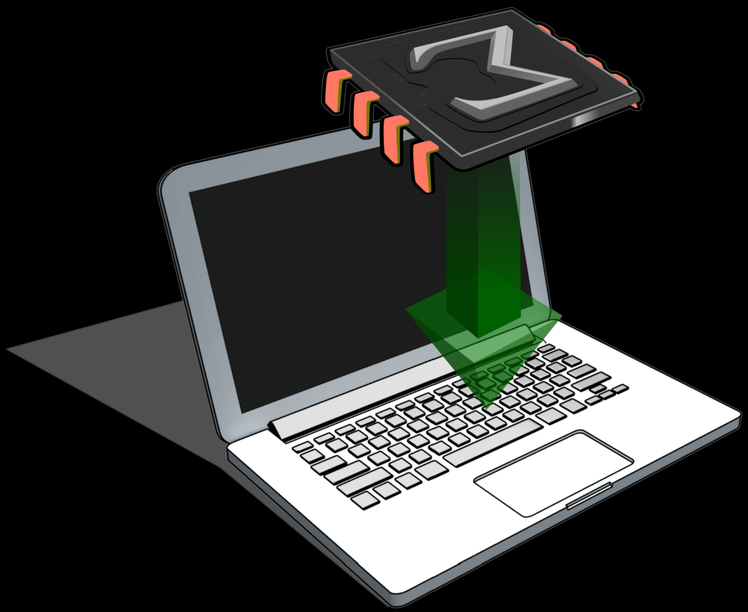
$5040+5040\varepsilon+2520\varepsilon^2+840\varepsilon^3+210\varepsilon^4+42\varepsilon^5+7\varepsilon^6+\varepsilon^7$ $\quad \approx$64 bits

FFT trading



Binary splitting exponentiation in time $O\left(\mathsf{M}(n)\,\frac{\log^2 n}{\log\log n}\right)$

Thank you !



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